Two results on $\mathcal{C}^\infty$-rings:
an $\mathbb{R}$-algebra morphism of $\mathcal{C}^\infty$-rings which is not a $\mathcal{C}^\infty$-morphism and von Neumann regular $\mathcal{C}^\infty$-rings are real closed

"$\mathcal{C}^\infty$ meets real closed" working group
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Context and motivation

The case for real closed rings I

Let

$$\mathcal{C}_{\mathrm{csa}} := \bigcup_{n \in \mathbb{N}_0} \{ f : \mathbb{R}^n \longrightarrow \mathbb{R} \mid f \text{ is continuous and } \mathbb{Z}\text{-semi-algebraic} \},$$

and if A is any set, then let $\mathcal{O}_A := \bigcup_{n \in \mathbb{N}_0} \{ f : A^n \longrightarrow A \mid f \text{ is a function} \}$.

Definition (2.1 in [Tre07])

A ring A is **real closed** if there exists a function $\Phi : \mathcal{C}_{\mathrm{csa}} \longrightarrow \mathcal{O}_A$ such that:

1. If $n \in \mathbb{N}_0$ and $f \in \mathcal{C}_{\mathrm{csa}}$ is n-ary, then $\Phi(f)$ is n-ary.
2. If $n \in \mathbb{N}$ and $f \in \mathcal{C}_{\mathrm{csa}}$ is the projection $\mathbb{R}^n \longrightarrow \mathbb{R}$ onto the i^{th}-coordinate, then $\Phi(f)$ is the projection $A^n \longrightarrow A$ onto the i^{th}-coordinate.
3. If $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $f \in \mathcal{C}_{\mathrm{csa}}$ is m-ary, and $g_1, \ldots, g_m \in \mathcal{C}_{\mathrm{csa}}$ are n-ary, then

$$\Phi(f \circ (g_1, \ldots, g_m)) = \Phi(f) \circ (\Phi(g_1), \ldots, \Phi(g_m)).$$

4. $\Phi(+_{\mathbb{R}}) = +_A$, $\Phi(\cdot_{\mathbb{R}}) = \cdot_A$, and $\Phi(-_{\mathbb{R}}) = -_A$, $\Phi(0_{\mathbb{R}}) = 0_A$, and $\Phi(1_{\mathbb{R}}) = 1_A$.

If A is a real closed ring, then the set $\Phi(\mathcal{C}_{\mathrm{csa}}) = \{\Phi(f) \mid f \in \mathcal{C}_{\mathrm{csa}}\}$ is a **real closed structure** for A.

A ring is real closed in the sense above if and only if it is a real closed ring in the sense of Niels Schwartz (this fact is implicit in [SM99], Chapters 7 and 12).

Context and motivation

The case for real closed rings II

Theorem (2.12 (c) in [Tre07])

Let A be a real closed ring. Then A has a unique real closed structure, that is, there exists a unique function $\Phi_A : \mathcal{C}_{csa} \longrightarrow \mathcal{O}_A$ satisfying all items 1 - 4 from the definition of the previous slide.

Proposition (2.16 in [Tre07])

Let $F : A \longrightarrow B$ be a ring homomorphism of real closed rings. Then F preserves the real closed structure, that is, for every $n \in \mathbb{N}_0$ and every n-ary function $f \in \mathcal{C}_{csa}$ the equality

$$F(\Phi_A(f)(a_1, \ldots, a_n)) = \Phi_B(f)(F(a_1), \ldots, F(a_n))$$

holds.

Question: Do the any of the above statements also hold for $\mathcal{C}^\infty$-rings?

Partial answer: The second statement is false for $\mathcal{C}^\infty$-rings as there exists a counterexample; this is Remark 4.40 in [CR13]. I don't know if the first statement holds for $\mathcal{C}^\infty$-rings.

The counterexample was constructed by Reichard in [Rei75] to answer a similar question posed by Malgrange (Remark 2.4 in [Mal66]) in the context of differentiable algebras. This counterexample is also mentioned in the context of $\mathcal{C}^\infty$-differentiable spaces, see the line above Example 2.37 in [NGSdS03].

The counterexample

Choice of data

Fix the following notation:

- $\mathbb{R}\{x\} \subseteq \mathbb{R}[[x]]$ is the subring of convergent power series in the variable x over $\mathbb{R}$.
- $C^\infty(\mathbb{R})_0$ is the ring of germs of smooth functions $\mathbb{R} \longrightarrow \mathbb{R}$ at 0.
- $C^\omega(\mathbb{R})_0 \subseteq C^\infty(\mathbb{R})_0$ is the subring of germs of analytic functions $\mathbb{R} \longrightarrow \mathbb{R}$ at 0.
- $T : C^\infty(\mathbb{R})_0 \twoheadrightarrow \mathbb{R}[[x]]$ is the Taylor series map at 0, that is, if $[f]$ is the germ at 0 of $f \in C^\infty(\mathbb{R})$, then $T([f]) := \sum_{n=0}^\infty \frac{f^{(n)}(x)}{n!} x^n$.

Fact (Pages 18 and 31 in [MR91])

The rings $\mathbb{R}[[x]]$ and $C^\infty(\mathbb{R})_0$ are $\mathcal{C}^\infty$-rings $(\mathbb{R}[[x]], \Phi_1)$ and $(C^\infty(\mathbb{R})_0, \Phi_2)$. In particular:

- *If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function and $a = \sum_{n=0}^\infty a_n x^n \in \mathbb{R}[[x]]$, then*

$$\Phi_1(f)(a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a_0)}{n!}(a - a_0)^n.$$

- *If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function and $[g] \in C^\infty(\mathbb{R})_0$, then*

$$\Phi_2(f)([g]) = [f \circ g].$$

- *T preserves the $\mathcal{C}^\infty$-structure, that is, T is a $\mathcal{C}^\infty$-morphism $T : (C^\infty(\mathbb{R})_0, \Phi_2) \twoheadrightarrow (\mathbb{R}[[x]], \Phi_1)$.*

The counterexample

Reichard's theorem

Theorem (Satz 2 in [Rei75])

There exists an $\mathbb{R}$-algebra morphism $F : \mathbb{R}[[x]] \longrightarrow C^\infty(\mathbb{R})_\circ$ such that:

1. *F is an $\mathbb{R}$-algebra cross-section of T, that is, $T \circ F : \mathbb{R}[[x]] \longrightarrow \mathbb{R}[[x]]$ is the identity map.*
2. *F restricts to the canonical $\mathbb{R}$-algebra isomorphism $F_\circ : \mathbb{R}\{x\} \xrightarrow{\cong} C^\omega(\mathbb{R})_\circ$. In particular, $F(x) = [\mathrm{id}_\mathbb{R}]$, where $\mathrm{id}_\mathbb{R}$ is the identity map $\mathbb{R} \longrightarrow \mathbb{R}$.*

Sketch of proof

Define M to be the set of pairs (A, f) such that:

- A is an $\mathbb{R}$-algebra with $\mathbb{R}\{x\} \subseteq A \subseteq \mathbb{R}[[x]]$.
- $f : A \longrightarrow C^\infty(\mathbb{R})_\circ$ is an $\mathbb{R}$-algebra morphism.
- f restricts to the canonical isomorphism $F_\circ : \mathbb{R}\{x\} \xrightarrow{\cong} C^\omega(\mathbb{R})_\circ$.

Define a partial order $\sqsubseteq$ on M by setting $(A, f) \sqsubseteq (B, g) \overset{\text{def}}{\iff} A \subseteq B$ and $g{\restriction_A} = f$. The set M is non-empty since $(\mathbb{R}\{x\}, F_\circ) \in M$, therefore the poset $(M, \sqsubseteq)$ has a maximal element (A, F) by Zorn's lemma. Assume for contradiction that $A \neq \mathbb{R}[[x]]$. Choose $b \in \mathbb{R}[[x]] \setminus A$. Then there exists an $\mathbb{R}$-algebra morphism $G : A[b] \longrightarrow C^\infty(\mathbb{R})_\circ$ which extends F, yielding a contradiction to maximality of (A, F) in $(M, \sqsubseteq)$. □

The counterexample Reichard's $\mathbb{R}$-algebra morphism F is not a $\mathcal{C}^\infty$-morphism

Observation (cf. Remark 4.40 in [CR13])

The $\mathbb{R}$-algebra morphism $F : \mathbb{R}[[x]] \longrightarrow C^\infty(\mathbb{R})_0$ from Reichard's theorem does not preserve the $\mathcal{C}^\infty$-structure, that it, it is not a $\mathcal{C}^\infty$-morphism $F : (\mathbb{R}[[x]], \Phi_1) \longrightarrow (C^\infty(\mathbb{R})_0, \Phi_2)$.

Proof

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be any smooth function which is flat at 0 (that is, such that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}_0$) and such that $[f] \neq [0]$ in $C^\infty(\mathbb{R})_0$. An example of such function is the one defined by $f(0) = 0$ and $f(x) := e^{\frac{-1}{x^2}}$ if $x \neq 0$.

Let $\mathrm{id}_\mathbb{R}$ be the identity map $\mathbb{R} \longrightarrow \mathbb{R}$. On the one hand, $F(x) = [\mathrm{id}_\mathbb{R}]$ by item 2 in Reichard's theorem, therefore

$$\Phi_2(f)(F(x)) = \Phi_2(f)([\mathrm{id}_\mathbb{R}]) = [f \circ \mathrm{id}_\mathbb{R}] = [f].$$

On the other hand, $\Phi_1(f)(x) = \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!}(x-0)^n = 0$ since f is flat at 0, therefore

$$F(\Phi_1(f)(x)) = F(0) = [0],$$

and thus

$$F(\Phi_1(f)(x)) \neq \Phi_2(f)(F(x))$$

by choice of f. □

Follow-up questions

Question: Does every $\mathcal{C}^\infty$-ring have a unique $\mathcal{C}^\infty$-structure?

Let $\mathcal{C}^\infty$**-RING** be the category of $\mathcal{C}^\infty$-rings with $\mathcal{C}^\infty$-morphisms and $\mathbb{R}$**-Alg** be the category of $\mathbb{R}$-algebras with $\mathbb{R}$-algebra morphisms. The previous counterexample implies that the forgetful functor $U : \mathcal{C}^\infty\textbf{-RING} \longrightarrow \mathbb{R}\textbf{-Alg}$ is not full.

However, there exist full subcategories $\mathscr{C} \subseteq \mathcal{C}^\infty$**-RING** such that the restriction of U to $\mathscr{C}$ is a full functor. For example, one can take $\mathscr{C}$ to be the full subcategory of $\mathcal{C}^\infty$**-RING** whose objects are either of the following:

- $\mathcal{C}^\infty(M)$ where M is a smooth manifold (Theorem 2.8 and Corollary 3.7 in [MR91]).
- Formal algebras, that is, factor rings of formal power series with finitely many variables over $\mathbb{R}$ (Proposition 3.20 in [MR91]).
- Weil algebras (Corollary 3.19 in [MR91]).
- Near-point determined $\mathcal{C}^\infty$-rings (Definition 4.1 (b) in [MR91] and Proposition 8 in [Bor15]).

Question: Is there a largest full subcategory $\mathscr{C} \subseteq \mathcal{C}^\infty$**-RING** such that the restriction of U to $\mathscr{C}$ is a full functor? If not, is there an intrinsic characterization of the categories $\mathscr{C}$ that satisfy this property?

Proof ingredients

Theorem (Theorem 2.10 in [MR86] and page 42 in [MR91])

(The underlying field of) Any $\mathcal{C}^\infty$-field K is a real closed field.

Sketch of proof

The proof heavily relies on the representation of $\mathcal{C}^\infty$-rings as quotients by ideals of filtered colimits of $\mathcal{C}^\infty$-rings of the form $C^\infty(\mathbb{R}^n)$. In particular, $K \cong C^\infty(\mathbb{R}^E)/I$ as $\mathcal{C}^\infty$-rings for some set E and some maximal ideal $I \subseteq C^\infty(\mathbb{R}^E)$, where

$$C^\infty(\mathbb{R}^E) := \varinjlim\{C^\infty(\mathbb{R}^D) \mid D \subseteq E \text{ finite}\}.$$

Maximality of I implies that $f/I > 0 \overset{\text{def}}{\iff} \exists g \in I, \forall x \in Z(g), f(x) > 0$ is a total order on $C^\infty(\mathbb{R}^E)/I$ turning it into a totally ordered field. Smooth Tietze implies that that positive elements in this total order are squares.

Let $p(t) \in K[t]$ be a monic and irreducible polynomial of odd degree. Let

$$f(x,t) = t^n + f_1(x)t^{n-1} + \ldots + f_n(x) \in C^\infty(\mathbb{R}^E)[t]$$

be any polynomial such that $p(t) = t^n + (f_1(x)/I)t^{n-1} + \ldots + (f_n(x)/I)$. Then there exists a smooth function $r : U \longrightarrow \mathbb{R}$ for some open $U \subseteq \mathbb{R}^D$ and a finite set $D \subseteq E$ such that $r(x)$ is the first root of $f(x,t) \in \mathbb{R}[t]$ (which exists since $\mathbb{R}$ is real closed). Then "r/I" is a root of $p(t)$. $\square$

Proof ingredients

Integrally ringed spaces

Definition (Definition 2 in [Sch90a])

A ringed space $X = (X, \mathcal{O}_X)$ is an *integrally ringed space* if each stalk is an integral domain. If $X = (X, \mathcal{O}_X)$, $Y = (Y, \mathcal{O}_Y)$ are integrally ringed spaces and if $f = (f, f^{\#}) : X \longrightarrow Y$ is a morphism in the category of ringed spaces, then f is a morphism of integrally ringed spaces when for all $x \in X$ the homomorphism $f^{\#}_X : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$ of stalks is a monomorphism. Write **IntRingSp** for the resulting category.

Let A be a ring. Write $\mathrm{Spec}^{\mathrm{inv}}(A)$ for $\mathrm{Spec}(A)$ equipped with the inverse spectral topology, so that the sets $V(a_1, \ldots, a_n) := \{\mathfrak{p} \in \mathrm{Spec}(A) \mid a_1, \ldots, a_n \in \mathfrak{p}\}$ form a basis of quasi-compact opens for the topology on $\mathrm{Spec}^{\mathrm{inv}}(A)$.

Theorem (Satz 3 and Satz 4 in [Sch90a])

Let A be a ring.

1. *Let* $\mathcal{O}_\circ(V(a_1, \ldots, a_n)) := A/\sqrt{(a_1, \ldots, a_n)}$ *and* $\mathcal{O}_\circ(V(a_1, \ldots, a_n) \subseteq V(b_1, \ldots, b_m))$ *be the canonical surjection* $A/\sqrt{(b_1, \ldots, b_m)} \twoheadrightarrow A/\sqrt{(a_1, \ldots, a_n)}$. *Then* $\mathcal{O}_\circ$ *is a presheaf on (the basis of)* $\mathrm{Spec}^{\mathrm{inv}}(A)$.
2. *Let* $\mathcal{O}$ *be the unique sheaf on* $\mathrm{Spec}^{\mathrm{inv}}(A)$ *associated to* $\mathcal{O}_\circ$. *Then* $\mathrm{Spec}^{\mathrm{inv}}(A) = (\mathrm{Spec}^{\mathrm{inv}}(A), \mathcal{O})$ *is an integrally ringed space.*
3. *The presheaf* $\mathcal{O}_\circ$ *is a sheaf if and only if sums of radical ideals of A are radical ideals. This condition holds, for example, if A is a real closed ring (Section 3 in [Sch86]).*

Proof ingredients

A universal property and its main application

Theorem (Satz 16 in [Sch90a])

*Let A be a ring and $\rho_A : A \longrightarrow \rho(A)$ be its real closure (Chapter I in [Sch89]). Then the pair $(\mathrm{Spec}^{\mathrm{inv}}(\rho(A)), \mathrm{Spec}^{\mathrm{inv}}(\rho_A))$ has the following universal property: for every $X \in$ **IntRingSp** with real closed stalks and every **IntRingSp**-morphism $f : X \longrightarrow \mathrm{Spec}^{\mathrm{inv}}(A)$ there exists a unique **IntRingSp**-morphism $\bar{f} : X \longrightarrow \mathrm{Spec}^{\mathrm{inv}}(\rho(A))$ such that $\mathrm{Spec}^{\mathrm{inv}}(\rho_A) \circ \bar{f} = f$.*

Theorem (Satz 17 in [Sch90a])

Let X be an integrally ringed space with real closed stalks. Then $A := \Gamma(X)$ is a real closed ring.

Sketch of proof

The identity map $A \longrightarrow A$ induces a **IntRingSp**-morphism $f : X \longrightarrow \mathrm{Spec}^{\mathrm{inv}}(A)$ (Satz 7 in [Sch90b]), therefore by Satz 16 in [Sch90a] there exists a unique **IntRingSp**-morphism $\bar{f} : X \longrightarrow \mathrm{Spec}^{\mathrm{inv}}(\rho(A))$ such that $\mathrm{Spec}^{\mathrm{inv}}(\rho_A) \circ \bar{f} = f$. Applying the functor Γ : **IntRingSp** $\longrightarrow$ **Ring** yields a commutative diagram of rings and ring homomorphisms

$$\begin{tikzcd} & \Gamma(X) = A \\ \Gamma(\mathrm{Spec}^{\mathrm{inv}}(A)) \arrow[ur, "\Gamma(f)"] \arrow[r, "\Gamma(\mathrm{Spec}^{\mathrm{inv}}(\rho_A))"'] & \Gamma(\mathrm{Spec}^{\mathrm{inv}}(\rho(A))) = \rho(A) \arrow[u, "\Gamma(\bar{f})"'] \end{tikzcd}$$

$\Gamma(f)$ is an isomorphism, therefore $\Gamma(\bar{f})$ is surjective. Since A is reduced, A is real closed. $\square$

The proof

Proposition (P.P.)

(The underlying ring of) Any von Neumann regular $\mathcal{C}^{\infty}$-ring A is a real closed ring. In particular, the support map (of the underlying ring) $\mathrm{supp}_A : \mathrm{Sper}(A) \longrightarrow \mathrm{Spec}(A)$ *is a homeomorphism.*

Proof

Since A is von Neumann regular, $\mathrm{Spec}(A) = \mathrm{Spec}^{\max}(A)$. Recall that if B is any $\mathcal{C}^{\infty}$-ring:

1. For every ring-theoretic ideal $I \subseteq B$, the ring B/I has a canonical $\mathcal{C}^{\infty}$-structure and the map $B \twoheadrightarrow B/I$ is a $\mathcal{C}^{\infty}$-morphism (Proposition 1.2 in [MR91]).
2. $\mathfrak{p} \in \mathrm{Spec}(B)$ is a $\mathcal{C}^{\infty}$-radical prime ideal (p. 329 in [MR86], and p. 2025 and Section 4 in [BM23]) if and only if the $\mathcal{C}^{\infty}$-ring $B/\mathfrak{p}$ is a $\mathcal{C}^{\infty}$-subring of a $\mathcal{C}^{\infty}$-field (Proposition 2.5 in [MR86]). The set $\mathrm{Spec}^{\infty}(B)$ of all $\mathcal{C}^{\infty}$-radical prime ideals of B is a proconstructible subset of $\mathrm{Spec}(B)$ (Section 1 in [MvQR87] and Section 4 in [BM23]).

It follows that

$$X := \mathrm{Spec}(A) = \mathrm{Spec}^{\max}(A) = \mathrm{Spec}^{\infty}(A). \qquad (*)$$

as topological spaces; see also Theorem 3.11 2) in [BM23]. Let $\mathcal{O}_X$ be the Zariski sheaf on X. Then $(*)$ and the fact that every $\mathcal{C}^{\infty}$-field is real closed field together imply that $X = (X, \mathcal{O}_X)$ is an integrally ringed space with real closed stalks. Classical commutative algebra yields $A \cong \Gamma(X)$ as rings, therefore A is a real closed ring by Satz 17 in [Sch90a]. The statement about the support map being a homeomorphism is true for any real closed ring (Theorem 3.10 in [Sch89, Chapter I] and Theorem 12.4 (d) in [SM99]). □

Follow-up questions

Question: Are there any other $\mathcal{C}^\infty$-rings which are real closed?

Most likely the answer is yes. A possible first approach to this question is characterizing those valuation $\mathcal{C}^\infty$-rings which are real closed. Recall that if A is a local $\mathcal{C}^\infty$-ring, then A is Henselian with real closed residue field (Corollary 2.11 in [MR86]), therefore a sensible conjecture would be that if V is a valuation $\mathcal{C}^\infty$-ring, then

$$V \text{ is real closed} \iff \text{the value group of } (\mathrm{qf}(V), V) \text{ is divisible.}$$

This should follow from the Ax-Kochen-Ershov theorem for Henselian valued fields of equicharacteristic 0.

Question: Does the Hahn power series construction $R[[\Gamma]] := R[[x^{\Gamma^{\geqslant 0}}]]$ (R a field, Γ a totally ordered abelian group) yield examples of valuation $\mathcal{C}^\infty$-rings? More concretely, if $R = (R, \Phi)$ is a $\mathcal{C}^\infty$-field, can a similar definition of the $\mathcal{C}^\infty$-structure on $\mathbb{R}[[x]]$ be used to equip $R[[\Gamma]]$ with a $\mathcal{C}^\infty$-structure Φ^*? For example, if $f \in \mathcal{C}^\infty$ is unary, and $a = \sum_{n=0}^\infty a_n x^n \in R[[\Gamma]]$, then

$$\Phi^*(f)(a) := \sum_{n=0}^{\infty} \frac{(\Phi(f^{(n)})(a_0)}{n!}(a - a_0)^n ?$$

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