Valuation $\mathcal{C}^\infty$-rings:
first steps

“$\mathcal{C}^\infty$ meets real closed” working group
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Basic notions

Fix the following notation and conventions:

- If A is any ring, then $A^\times$ is its multiplicative group of units.
- If A is a domain, then $\mathrm{qf}(A)$ is its quotient field, i.e., $\mathrm{qf}(A)$ is the field obtained as the localization of A by the multiplicate subset $A \setminus \{0\}$ in the category of rings. Regard A as a subring of $\mathrm{qf}(A)$ in the canonical way.
- If $A \subseteq K$ is a subring of a field K, then regard $\mathrm{qf}(A)$ as a subfield of K in the canonical way by the universal property of $\mathrm{qf}(A)$. In particular, $A \subseteq \mathrm{qf}(A) \subseteq K$.

If $A \subseteq K$ is a subring of a field K, then A is a *valuation ring of K* if for all $b \in K^\times$, either $b \in A$ or $b^{-1} \in A$. An arbitrary ring A is a *valuation ring* if A is a domain and a valuation ring of $\mathrm{qf}(A)$. Note that if $A \subseteq K$ is a valuation ring of a field K, then $\mathrm{qf}(A) = K$: if $b \in K^\times$, then either $b \in A$, in which case $b \in \mathrm{qf}(A)$, or $b^{-1} \in A$, in which case $b \in \mathrm{qf}(A)$ also follows.

If $A \subseteq K$ is a valuation ring of a field K, then the pair (K, A) is called a *valued field*. For a valued field (K, A), the quotient of multiplicative groups $K^\times / A^\times$ is called the *value group of* (K, A) and it has the structure of a totally ordered abelian group by setting $b/A^\times \leqslant c/A^\times$ if and only if $cb^{-1} \in A$. Valuation rings are local rings. The *residue field* of a valued field (K, A) is $A/\mathfrak{m}_A$, where $\mathfrak{m}_A$ is the unique maximal ideal of A. A standard reference for valued fields is [EP05].

The Ax-Kochen-Ershov theorem

A valued field (K, A) is *Henselian* if Hensel's lemma holds for $(A, \mathfrak{m}_A)$; for equivalent characterizations of Henselian valued fields see [EP05, Theorem 4.1.3]. A valued field (K, A) is said to be of *equicharacteristic* 0 if both K and $A/\mathfrak{m}_A$ are fields of characteristic 0.

Theorem (Ax-Kochen-Ershov theorem)

Let (K, A) and (L, B) be Henselian valued fields of equicharacteristic 0. The following statements are equivalent:

1. $(K, A) \equiv (L, B)$ *in the language* $\mathscr{L}^{\mathrm{ring}} \cup \{P\}$, *where* $\mathscr{L}^{\mathrm{ring}} := \{+, -, \cdot, 0, 1\}$ *and P is a unary predicate interpreted as the valuation ring of the valued field.*
2. $A/\mathfrak{m}_A \equiv B/\mathfrak{m}_B$ *in the language* $\mathscr{L}^{\mathrm{ring}}$ *and* $K^\times/A^\times \equiv L^\times/B^\times$ *in the language* $\{+, -, 0, \leqslant\}$ *of ordered groups.*

Proof

Folklore; see for instance [vdD14, Theorem 5.1]. Note that this theorem is usually phrased for valued fields regarded as three-sorted structures (e.g. [vdD14, Theorem 5.11], [KC90, Theorem 5.4.12], or [d'E23, Theorem 1.15]). The statement of this theorem follows from the three-sorted version due to the fact that there exists a uniform bi-interpretation without parameters from the theory of valued fields in the language $\mathscr{L}^{\mathrm{ring}} \cup \{P\}$ to the theory of valued fields in the three-sorted language, therefore $(K, A) \equiv (L, B)$ if and only if their corresponding three-sorted structures are elementary equivalent in the three-sorted language. $\square$

Hahn series

Definition

Let $\boldsymbol{k}$ be a field and Γ be a totally ordered abelian group. Define $\boldsymbol{k}((\Gamma)) := \boldsymbol{k}((x^\Gamma))$ to be the set of formal series $a = \sum a_\gamma x^\gamma := \sum_{\gamma\in\Gamma} a_\gamma x^\gamma$ such that $\operatorname{supp}(a) := \{\gamma \in \Gamma \mid a_\gamma \neq 0\}$ is a well-ordered subset of Γ.

Theorem (Exercise 3.5.6 and Remark 4.1.8 in [EP05])

Let $\boldsymbol{k}$ be a field and Γ be a totally ordered abelian group.

1. *The set $\boldsymbol{k}((\Gamma))$ endowed with the operations of pointwise addition and Cauchy product of formal series*

$$\sum a_\gamma x^\gamma + \sum b_\gamma x^\gamma := \sum (a_\gamma + b_\gamma) x^\gamma$$

   and

$$\left(\sum a_\gamma x^\gamma\right)\left(\sum b_\gamma x^\gamma\right) := \sum_{\gamma\in\Gamma}\sum_{\delta+\epsilon=\gamma}(a_\delta b_\epsilon)x^\gamma,$$

   respectively, is a field called Hahn field.
2. *The set $\boldsymbol{k}[[\Gamma]] := \boldsymbol{k}((x^{\Gamma^{\geqslant 0}})) \subseteq \boldsymbol{k}((\Gamma))$ is a valuation ring of $\boldsymbol{k}((\Gamma))$.*
3. *The value group of $(\boldsymbol{k}((\Gamma)), \boldsymbol{k}[[\Gamma]])$ is Γ and its residue field is $\boldsymbol{k}$.*
4. *The valued field $(\boldsymbol{k}((\Gamma)), \boldsymbol{k}[[\Gamma]])$ is Henselian.*

Equivalent characterizations of real closed valuation rings

Definition

A ring is a *real closed valuation ring* if it is both a valuation ring and a real closed ring.

Theorem (Theorem 3.1.4 in [Pie24])

Let A be a ring. The following are equivalent:

1. *A is a real closed valuation ring.*
2. *A is a domain, $\mathrm{qf}(A)$ is a real closed field, and A is convex in $\mathrm{qf}(A)$.*
3. *A is a valuation ring, and both $\mathrm{qf}(A)$ and $A/\mathfrak{m}_A$ are real closed fields.*
4. *A is a totally ordered domain which satisfies the intermediate value property for polynomials in one variable.*
5. *A is a totally ordered domain which satisfies the following conditions:*
   a) *For all $a, b \in A$, if $0 < a < b$, then there exists $c \in A$ such that $bc = a$.*
   b) *Every positive element has a square root.*
   c) *Every monic polynomial of odd degree has a root.*
6. *A is a local real closed SV-ring of rank 1.*

Proposition (Proposition 2.1.7 (a) and Proposition 2.2.4 in [KS22])

Let K be a totally ordered field. Then any convex subring of K is a valuation ring of K. In particular, if $B \subseteq K$ is any subring, then the convex hull of B in K is a convex valuation ring of K.

Real closed valued fields

Definition

A *real closed valued field* is a valued field (K, A) where A is a real closed valuation ring (equivalently, K is a real closed field and A is convex in K).

Example

Let $\boldsymbol{k}$ be a real closed field and Γ be a divisible totally ordered abelian group. Then $(\boldsymbol{k}((\Gamma)), \boldsymbol{k}[[\Gamma]])$ is a real closed valued field. The converse also holds; see [AvdDvdH17, Example in p. 177].

Corollary

Let (K, A) be a Henselian valued field of equicharacteristic 0. If the residue field $\boldsymbol{k} := A/\mathfrak{m}_A$ is a real closed field and the value group $\Gamma := K^\times/A^\times$ is divisible, then K is a real closed field and A is a convex in K.

Sketch of proof

Since $(\boldsymbol{k}((\Gamma)), \boldsymbol{k}[[\Gamma]])$ is a Henselian valued field, the Ax-Kochen-Ershov theorem implies that $(K, A) \equiv (\boldsymbol{k}((\Gamma)), \boldsymbol{k}[[\Gamma]])$ in the language of $\mathscr{L}^{\text{ring}} \cup \{P\}$. Since the theory of real closed valued fields is axiomatizable in the language $\mathscr{L}^{\text{ring}} \cup \{P\}$ and $(\boldsymbol{k}((\Gamma)), \boldsymbol{k}[[\Gamma]])$ is a real closed valued field, the statement follows. □

Two notions of valuation rings in the category of $\mathcal{C}^\infty$-rings

Fix the following notation and conventions:

- A *$\mathcal{C}^\infty$-reduced $\mathcal{C}^\infty$-domain* is a $\mathcal{C}^\infty$-ring $\mathcal{C}^\infty$-isomorphic to a $\mathcal{C}^\infty$-subring of a $\mathcal{C}^\infty$-field.
- If A is a $\mathcal{C}^\infty$-reduced $\mathcal{C}^\infty$-domain, then $k(A)$ is its quotient field in the category of $\mathcal{C}^\infty$-rings, i.e., $k(A)$ is the $\mathcal{C}^\infty$-field obtained as the localization of A by the multiplicative subset $A \setminus \{0\}$ in the category of $\mathcal{C}^\infty$-rings. Regard A as a $\mathcal{C}^\infty$-subring of $k(A)$ in the canonical way, and in particular $A \subseteq \mathrm{qf}(A) \subseteq k(A)$ by the universal property of $\mathrm{qf}(A)$ in the category of rings.

Definition

Let A be a $\mathcal{C}^\infty$-ring.

1. A is a *valuation $\mathcal{C}^\infty$-ring* if it(s underlying ring) is a valuation ring.
2. A is a *$\mathcal{C}^\infty$-valuation $\mathcal{C}^\infty$-ring* if A is a $\mathcal{C}^\infty$-reduced $\mathcal{C}^\infty$-ring and (its underlying ring is) a valuation ring of $k(A)$.

Remark

Let A be a $\mathcal{C}^\infty$-ring.

1. If A is a valuation $\mathcal{C}^\infty$-ring, then in particular A is a $\mathcal{C}^\infty$-domain, but it might not be $\mathcal{C}^\infty$-reduced. For example, $\mathbb{R}[[x]]$ $(= \mathbb{R}[[\mathbb{Z}]])$ is a valuation $\mathcal{C}^\infty$-ring which is not $\mathcal{C}^\infty$-reduced (see example 1 in the proof of [BK18, Proposition 1]).
2. If A is a $\mathcal{C}^\infty$-valuation $\mathcal{C}^\infty$-ring, then A is a valuation $\mathcal{C}^\infty$-ring.

A characterization of real closed valuation $\mathcal{C}^\infty$-rings

Lemma

Let A be a valuation $\mathcal{C}^\infty$-ring. Then $(\mathrm{qf}(A), A)$ is a Henselian valued field of equicharacteristic 0 with real closed residue field.

Proof

Since A is a $\mathcal{C}^\infty$-ring, A is an $\mathbb{R}$-algebra, therefore $\mathrm{qf}(A)$ has characteristic 0. Since A is a local ring, it is Henselian by [MR86, Theorem 3.22] and its residue field is real closed because it is a $\mathcal{C}^\infty$-field ([MR91, Proposition 1.2]) and $\mathcal{C}^\infty$-fields are real closed ([MR86, Theorem 2.10]). □

Theorem (P.P.)

Let A be a valuation $\mathcal{C}^\infty$-ring. Then A is a real closed ring if and only if its value group $\Gamma := \mathrm{qf}(A)^\times / A^\times$ is divisible.

Sketch of proof

If A is a real closed ring, then $\mathrm{qf}(A)$ is a real closed field. In particular, since monic polynomials of odd degree have roots in $\mathrm{qf}(A)$, it follows that the multiplicative group $K^{>0}$ is divisible. Since the restriction of the surjective group homomorphism $\mathrm{qf}^\times(A) \twoheadrightarrow \Gamma$ to $K^{>0}$ is surjective, it follows that Γ is divisible. Suppose now that Γ is divisible. Then $(\mathrm{qf}(A), A)$ is a Henselian valued field of equicharacteristic 0 with real closed residue field and divisible value group, therefore by a previous corollary $\mathrm{qf}(A)$ is real closed and A is convex in $\mathrm{qf}(A)$, from which it follows that A is a real closed (valuation) ring. □

Constructing real closed valuation $\mathcal{C}^\infty$-rings

Proposition (P.P.)

Let $\mathcal{L}^\infty := \{f \mid f \in \mathcal{C}^\infty\}$ be the language of $\mathcal{C}^\infty$-rings. Let (K, Φ_K) be a proper elementary extension of the $\mathcal{C}^\infty$-field $(\mathbb{R}, \Phi_\mathbb{R})$ in the language $\mathcal{L}^\infty$. Then the convex hull of $\mathbb{R}$ in K,

$$\text{c.h.}_K(\mathbb{R}) := \{a \in K \mid \exists r \in \mathbb{R}^{\geqslant 0} \text{ such that } -r \leqslant a \leqslant r\},$$

is a real closed $\mathcal{C}^\infty$-valuation $\mathcal{C}^\infty$-ring which is not a field.

Proof

Since K is a real closed field and $A := \text{c.h.}_K(\mathbb{R})$ is a convex subring of K, A is a real closed valuation ring. Since K a proper elementary extension of the real closed field $\mathbb{R}$, K contains elements which are not bounded by $\mathbb{R}$, therefore $A \neq K$.

To show that A is a $\mathcal{C}^\infty$-ring it suffices to show that $\Phi_K(f)(\overline{a}) \in A$ for every n-ary $f \in \mathcal{C}^\infty$ ($n \in \mathbb{N}_0$) and every $\overline{a} \in A^n$, as in this case, setting $\Phi_A(f) := \Phi_K(f)\restriction_{A^n}$ yields a $\mathcal{C}^\infty$-structure on A. In this way, A becomes a $\mathcal{C}^\infty$-subring of the $\mathcal{C}^\infty$-field K.

Pick and n-ary function $f \in \mathcal{C}^\infty$ ($n \in \mathbb{N}_0$) and $\overline{a} := (a_1, \ldots, a_n) \in A^n$. Since $\overline{a} \in A^n$, there exist $r_1, \ldots, r_n \in \mathbb{R}^{\geqslant 0}$ such that $-r_i \leqslant a_i \leqslant r_i$ for all $i \in \{1, \ldots, n\}$. Because $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous, the image of the closed and bounded box $[-r_1, r_1] \times \ldots \times [-r_n, r_n] \subseteq \mathbb{R}^n$ under f is bounded.

Constructing real closed valuation $\mathcal{C}^\infty$-rings

Proof (continued)

In other words, there exists $s \in \mathbb{R}^{\geqslant 0}$ such that

$$\forall x_1 \dots x_n \in \mathbb{R} \left[\left(\bigwedge_{i=1}^{n} -r_i \leqslant x_i \leqslant r_i \right) \to -s \leqslant f(x_1, \dots, x_n) \leqslant s \right].$$

Recall that $\mathscr{L}^\infty$ contains constants for all elements in $\mathbb{R}$. In particular, since $\mathbb{R}$ is a real closed field, the above statement about $\mathbb{R}$ and f can be expressed by the $\mathscr{L}^\infty$-sentence φ

$$\forall x_1 \dots x_n \left[\exists y_1 \dots y_n \exists z_1 \dots z_n \left(\bigwedge_{i=1}^{n} x_i + r_i = y_i^2 \wedge r_i - x_i = z_i^2 \right) \to \exists v \exists w (f(x_1, \dots, x_n) + s = v^2 \wedge s - f(x_1, \dots, x_n) = w^2) \right].$$

Since $(\mathbb{R}, \Phi_{\mathbb{R}}) \models \varphi$ and $(\mathbb{R}, \Phi_{\mathbb{R}}) \prec (K, \Phi_K)$, it follows that $(K, \Phi_K) \models \varphi$. Since K is a real closed field and $\bar{a} \in A^n \subseteq K^n$ satisfies $-r_i \leqslant a_i \leqslant r_i$ for all $i \in \{1, \dots, n\}$, it follows that $(K, \Phi_K) \models -s \leqslant \Phi_K(f)(\bar{a}) \leqslant s$, therefore $\Phi_K(f)(\bar{a}) \in A$, as required.

In particular, (A, Φ_A) is a $\mathcal{C}^\infty$-subring of the $\mathcal{C}^\infty$-field (K, Φ_K), therefore it is a $\mathcal{C}^\infty$-reduced $\mathcal{C}^\infty$-ring. Since A is a valuation ring of K, it follows that $K = \mathrm{qf}(A)$, therefore $\mathrm{qf}(A)$ is a $\mathcal{C}^\infty$-field and thus $\mathrm{qf}(A) = K = k(A)$ by the universal property of the quotient field $k(A)$ of A in the category if $\mathcal{C}^\infty$-rings. Therefore A is a $\mathcal{C}^\infty$-valuation $\mathcal{C}^\infty$-ring, concluding the proof. $\square$

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